DS 122 Practice Midterm 2

Name:

BUID:

Problem A (10 points)

Consider the situation of parents who are expecting twins. So-called "identical" twins are twins with identical DNA (meaning they will be born the same sex), while "fraternal" twins are twins whose similarity at the DNA level is like any pair of siblings. The parents are curious about whether their twins are identical (but the babies are not born yet). They know that among the general population, about $1/3$ of twins are identical.

Part 1

(2 points) Define the two possible hypotheses corresponding to the two cases of interest to the parents.

Solution:

The twins are identical or the twins are not identical.

Part 2

(2 points) What is the prior probability of each hypothesis?

Solution:

$P(\text{Identical}) = 2/3$ and $P(\text{Not Identical}) = 1/3$

Part 3

(2 points) The parents then get an ultrasound and find out that the twins are the same birth sex. What is the likelihood of this outcome under each hypothesis?

Solution:

$P(D \mid \text{Identical}) = 1$ and $P(D \mid \text{Not Identical}) = 1/2.$

Part 4

(4 points) What is the posterior probability of each hypothesis? (That is, given knowledge of the ultrasound, what should the parents conclude)?

Solution:

$$P(\text{Identical} \mid D) = \frac{P(D \mid \text{Identical})P(\text{Identical})}{P(D \mid \text{Identical})P(\text{Identical}) + P(D \mid \text{Not Identical})P(\text{Not Identical})} = \frac{1/2 \cdot 2/3}{(1/2 \cdot 2/3) + (1 \cdot 1/3)} = \frac{1}{2}$$

Problem B (10 points)

You are given three dice: a six-sided die, a ten-sided die, and a twenty-sided die. You choose one of the dice at random, roll it, and observe that you get a nine.

Denote H_0 the hypothesis that you rolled the six-sided die, H_1 the hypothesis that you rolled the ten-sided die, and H_2 the hypothesis that you rolled the twenty-sided die. Fill out the following table.

hyp	P(H)	P(D|H)	P(H)P(D|H)	P(H|D)
H_0				
H_1				
H_2				

Solution:

hyp	P(H)	P(D|H)	P(H)P(D|H)	P(H|D)
H_0	1/3	0	0	0
H_1	1/3	1/10	1/30	2/3
H_2	1/3	1/20	1/60	1/3

Problem C (10 points)

You are playing Dungeons and Dragons. You are attacking 2d4 (meaning your attack is the sum of two four-sided dice), but you only hit your opponent if you exceed their defense roll which is 1d6 (meaning a single roll of a six-sided dice). You want to determine the probability you will hit your opponent. It turns out this is the probabilty of superiority of your dice roll over theirs. Breaking this down into two parts:

Part 1

(5 points) What is the probability distribution of the sum of two four-sided dice

Solution:

Every exact pair of rolls has a 1/16 chance of occuring (e.g. the probability of rolling a 3 on the first die and 4 on the second is 1/4 * 1/4 = 1/16). So we need to count how many ways each outcome can occur:

Outcome	Number of Ways	Probability
1	0	0
2	1 way (1,1)	1/16
3	2 ways (1,2 or 2,1)	2/16 = 1/8
4	3 ways (1,3 or 2,2 or 3,1)	3/16
5	4 ways (1,4 or 2,3 or 3,2 or 4,1)	4/16 = 1/4
6	3 ways (2,4 or 3,3 or 4,2)	3/16
7	2 ways (3,4 or 4,3)	2/16 = 1/8
8	1 ways (4,4)	1/16

Part 2

(5 points) What is the probability of superiority of of the sum of two four-sided dice over a single 6 sided dice?

Solution:

The probability of superiority is the sum of the probabilities of pairs of outcomes where the sum of two four-sided dice is greater than the 6 sided die. One option would be to list out all $7 * 6 = 42$ pairs of outcomes and sum their probabilities. We can do this slightly faster as follows:

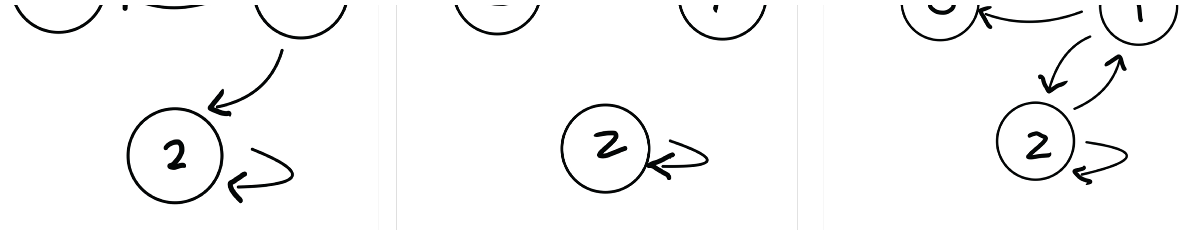
- An outcome of 8 is always superior, so we can add 1/16 to our probability of superiority.
- Same for 7 so we can add 1/8
- 6 is superior to all rolls other than 6, so we can add $(3/16) * (5/6)$
- 5 is superior to all rolls other than 5 or 6, so we can add $(1/4) * (4/6)$
- And so on

This give us:

Probability of superiority = $1/16 + 1/8 + (3/16) * (5/6) + (1/4) * (4/6) + (3/16) * (3/6) + (1/8) * (2/6) + (1/16) * (1/6) = 21/32$

Problem D (10 points)

Consider the three Markov Chains below. For each node, assume that all transitions out of that node are equally probable. So for a node with 3 outgoing transitions, the probabilities are each 1/3.



In this problem, "justify your answer" means you can explain in words, but if you rely on any calculations, you should show them.

Part 1

(3 pts) Write the transition matrices for the three Markov Chains. Order the column and row entries according to the numbers shown on the nodes.

A:

$$\begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$$

B:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C:

$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Part 2

(2 pts) Which of the three Markov Chains have at **least one** steady-state? Justify your answer.

All three because they are Markov Chains and every Markov Chain has at least one steady state

Part 3

(3 pts) Which of the three Markov Chains have a **unique** (ie, only one) steady-state? Justify your answer.

A has the steady state solution [0,0,1] and C has a steady state solution of [1/2,1/4,1/2]. In A, eveything dranins to node 2, and in C every state can reach every other state (i.e. the graphs are strongly connected).

Part 4

(2 pts) **For the Markov Chains that have unique steady-state,** which of them converge to that steady-state regardless of the starting state? Justify your answer.

A because the matrix is regular. P^2 has no zero entries.

Problem E (10 points)

For the Markov chain below, assume that the chain is in state

$$\begin{bmatrix} \frac{6}{11} \\ \frac{3}{11} \\ \frac{2}{11} \end{bmatrix}$$

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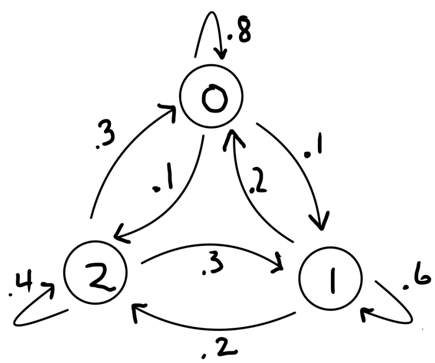

In this problem, "justify your answer" means you can explain in words, but if you rely on any calculations, you should show them.

Part 1

(5 pts) Is the chain in steady-state? Justify your answer.

Yes, $P\pi = \pi$:

$$\begin{bmatrix} 0.8 & 0.2 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} \frac{6}{11} \\ \frac{3}{11} \\ \frac{2}{11} \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{3}{11} \\ \frac{2}{11} \end{bmatrix}$$

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Part 2

(5 pts) Does the detailed balance condition hold? Justify your answer.

Yes, $\pi_i P_{ji} = \pi_j P_{ij}$ for all i, j. Specifically:

$$0.1 \cdot \frac{6}{11} = 0.2 \cdot \frac{3}{11}$$
$$0.1 \cdot \frac{6}{11} = 0.3 \cdot \frac{2}{11}$$

$$0.2 \cdot \frac{3}{11} = 0.3 \cdot \frac{2}{11}$$

Problem F (9 points)

Consider sampling a posterior distribution using MCMC.

Assume the desired posterior is the uniform distribution on $[0, 2]$ and 0 elsewhere, ie

$$p_T(x) = 1/2 \quad \text{for} \quad 0 \leq x < 2.$$

$$p_T(x) = 0 \quad \text{for} \quad x < 0, x \geq 2.$$

Assume the candidate distribution is uniform on $[0, 4]$, ie

$$p_C(x) = 1/4 \quad \text{for} \quad 0 \leq x < 4.$$

Recall that the Metropolis-Hastings rule is to accept a proposed transition from j to i with probability:

$$a_{ij} = \min\left(1, \frac{\pi_i H_{ji}}{\pi_j H_{ij}}\right)$$

Part 1

(3 points) Consider a transition from $x = 3$ to $x = 1/2$. What is the probability this transition will be accepted?

$$a_{1/2,3} = \min\left(1, \frac{1/2 * 1/4}{0 * 1/4}\right) = 1$$

Part 2

(3 points) Consider a transition from $x = 1/2$ to $x = 1$. What is the probability this transition will be accepted?

$$a_{1,1/2} = \min\left(1, \frac{1/2 * 1/4}{1/2 * 1/4}\right) = 1$$

Part 3

(3 points) Consider a transition from $x = 1$ to $x = 3$. What is the probability this transition will be accepted?

$$a_{3,1} = \min\left(1, \frac{0 * 1/4}{1/2 * 1/4}\right) = 0$$