Discussion 4

2 Oct 23

Problem 1

Say a population of elements is given as: {20,22,24}.

Use simple random sampling to write all possible samples of size two.

Calculate the variance of the population.

Solution

Random samples of size 2 will be: (20,22), (20,24), (22,24)

Mean is given by:

$$\mu = \frac{20 + 22 + 24}{3} = 22$$

Variance is given by:

$$\sigma^2 = \frac{400 + 484 + 576}{3} - 484 = \boxed{2.6}$$

Problem 2

A study from XYZ University claims that students in the week right before their exams, on average, study upto 50 hours a week with a standard deviation of 6 hours. Suppose the distribution of the number of hours is approximately normal.

1. Find the probability that a randomly selected student selected for this study will study less than 48 hours.
2. Find the probability that the mean of a random sample of 36 such students will be less than 48 hours.

Solution

Since the population is known to have a normal distribution

$$P(X < 48) = P(Z < \frac{48 - \mu}{\sigma}$$

$$= P(Z < \frac{48 - 50}{6})$$

$$= P(Z < -0.33) = \boxed{0.3707}$$

Solution

$$P(\overline{X} < 48) = P(Z < \frac{48 - \mu_X}{\sigma_X}$$

$$= P(Z < \frac{48 - 50}{1})$$

$$= P(Z < -2) = \boxed{0.0228}$$

Problem 3

Suppose you are conducting a survey to estimate the average income of a population. You collect data from a random sample of 100 individuals and find that the sample mean income is $50,000, with a standard deviation of $10,000. Calculate a 95% confidence interval for the population mean income.

Solution

The confidence interval for the population mean μ with a known population standard deviation σ is:

$$\text{Confidence Interval} = \bar{x} \pm Z\frac{\sigma}{\sqrt{n}}$$

Given:

- Sample mean income $\bar{x} = 50,000$
- Population standard deviation $\sigma = 10,000$
- Sample size $n = 100$
- Desired confidence level = 95%

We need to find the critical value Z for a 95% confidence interval. For a 95% confidence level, the critical value is approximately 1.96.

$$\text{Confidence Interval} = 50,000 \pm 1.96\frac{10,000}{\sqrt{100}}$$

$$= 50,000 \pm 1.96 \times 1,000$$

Now, calculate the upper and lower bounds of the confidence interval:

Upper Bound = $50,000 + 1.96 \times 1,000 \approx 51,960$

Lower Bound = $50,000 - 1.96 \times 1,000 \approx 48,040$

So, the 95% confidence interval for the population mean income is approximately $48,040$ to $51,960$.

Problem 4

A recent exam was given to a large group of students, and the scores followed a normal distribution with a mean (average) score of 75 and a standard deviation of 10.

1. Find the probability that a randomly selected student scored less than 60 on the exam.
2. Find the probability that the mean score of a random sample of 36 students will exceed 80.
3. Given that the top 5% of students received an award for their outstanding performance, determine the minimum score required to receive this award.

Solution

Part a

Z-score for a score of 60:

$$Z = \frac{X - \mu}{\sigma}$$

$$= \frac{60 - 75}{10} = -1.5$$

Using a Z-table, $P(Z < -1.5$ is approximately 0.0668.

So, the probability that a randomly selected student scored less than 60 on the exam is approximately 0.0668 or 6.68%.

Solution

Part b

Standard error of the sample mean:

$$SE = \frac{\sigma}{\sqrt{n}}$$

$$= \frac{10}{\sqrt{36}} = \frac{10}{6} = \frac{5}{3}$$

Z-score for a mean score of 80 using the standard error:

$$Z = \frac{\bar{X} - \mu}{SE}$$

$$= \frac{80 - 75}{\frac{5}{3}} = \frac{5}{\frac{5}{3}} = 3$$

Using a Z-table, $P(Z > 3)$ is approximately 0.0013.

So, the probability that the mean score of a random sample of 36 students will exceed 80 is approximately 0.0013 or 0.13%.

Solution

Part c

Z-score corresponding to the 95th percentile (since the top 5% corresponds to the 95th percentile).

The Z-score for the 95th percentile is approximately 1.645.

$$Z = \frac{X - \mu}{\sigma}$$

$$1.645 = \frac{X - 75}{10}$$

Now:

$$X = 1.645 \cdot 10 + 75 = 16.45 + 75 \approx 91.45$$

So, the minimum score required to receive an award for outstanding performance is approximately 91.45.

Problem 5

A tech company is interested in estimating the average battery life (in hours) of their latest smartphone model. They randomly select 40 smartphones from a production batch and record their battery life. The sample mean battery life is found to be 28 hours, and the sample standard deviation is 4 hours.

Assuming the battery life follows a normal distribution, find the 90% confidence interval for the true mean battery life of the smartphone.

Solution

Plug in the values:

- Sample Mean $\bar{X}$: 28 hours
- Sample Standard Deviation S: 4 hours
- Sample Size n: 40
- Critical Value Z: 1.645 (for 90% confidence)

Confidence interval:

$$\text{Confidence Interval} = \text{Sample Mean} \pm \left(\text{Critical Value} \times \frac{\text{Sample Standard Deviation}}{\sqrt{\text{Sample Size}}} \right)$$

$$= 28 \pm \left(1.645 \times \frac{4}{\sqrt{40}} \right)$$

$$= 28 \pm \left(1.645 \times \frac{4}{\sqrt{40}} \right) \approx 28 \pm 1.302$$

So, the 90% confidence interval for the true mean battery life of the smartphone is approximately $(26.698, 29.302)$ hours.