Extra Question. Suppose $\{x^{(1)}, x^{(2)} \ldots, x^{(n)}\}$ is an i.i.d. sample from the exponential distribution with rate λ. Consider the following estimator for $\theta = \frac{1}{\lambda}$:

$$\hat{\theta} = \frac{1}{n+1} \sum_{i=1}^{n} x^{(i)}.$$

a. Find the bias of $\hat{\theta}$.

b. Find the variance of $\hat{\theta}$.

c. Find the mean squared error of $\hat{\theta}$.

Solution to the extra question from the lecture on estimators.

$$\{X^{(1)}, X^{(2)}, \dots, X^{(n)}\} \overset{i.i.d.}{\sim} Exp(\lambda)$$

$\theta = \frac{1}{\lambda} \leftarrow$ the population mean

$$\hat{\theta} = \frac{1}{n+1} \sum_{i=1}^{n} X^{(i)}$$

a) $\text{bias}(\hat{\theta}) = E[\hat{\theta}] - \theta =$

$$= E\left[\frac{1}{n+1} \sum_{i=1}^{n} X^{(i)}\right] - \theta \overset{LOE}{=}$$

$$= \frac{1}{n+1} \sum_{i=1}^{n} E[X^{(i)}] - \theta \overset{\text{Exp. distr.}}{=}$$

$$= \frac{1}{n+1} \sum_{i=1}^{n} \theta - \theta =$$

$$= \frac{n}{n+1}\theta - \theta =$$

$$= \frac{n\theta - (n+1)\theta}{n+1} =$$

$$= -\frac{\theta}{n+1}.$$

b) $Var(\hat{\theta}) = Var\left(\frac{1}{n+1}\sum_{i=1}^{n} x^{(i)}\right) =$

i.i.d. sample

$$= \frac{1}{(n+1)^2} Var\left(\sum_{i=1}^{n} x^{(i)}\right) =$$

$$= \frac{1}{(n+1)^2} \sum_{i=1}^{n} Var\left(x^{(i)}\right) =$$

$$= \frac{n}{(n+1)^2} Var\left(x^{(1)}\right)$$

Exp. distr.

$$= \frac{n}{(n+1)^2} \cdot \theta^2 .$$

pop. mean $\theta = \frac{1}{\lambda}$

pop. variance $\theta^2 = \frac{1}{\lambda^2}$

c) From a) we know that $bias(\hat{\theta}) = -\frac{\theta}{n+1}$.

From b) we know that $Var(\hat{\theta}) = \frac{n\theta^2}{(n+1)^2}$

Thus, $MSE(\hat{\theta}) = bias(\hat{\theta})^2 + Var(\hat{\theta}) =$

$$= \left(-\frac{\theta}{n+1}\right)^2 + \frac{n\theta^2}{(n+1)^2} =$$

$$= \frac{\theta^2}{(n+1)^2} + \frac{n\theta^2}{(n+1)^2} = \frac{\cancel{(n+1)}\theta^2}{(n+1)^{\cancel{2}}} = \frac{\theta^2}{n+1}.$$