# DS 122 Homework 1 Analytical Section

**Xiang Fu**
xfu@bu.edu
Boston University Faculty of Computing & Data Sciences

# Contents

# 1 Problem A

## 1.1 Question

Consider a wooden cube with painted faces that is sawed up into 27 smaller equal-sized cubes. If one of these small cubes is chosen at random, what is the probability that it has exactly 3 painted faces?

## 1.2 Answer

Essentially, we can first visualize this as 3 layers of 3x3 grids of cubes.

Also, only the corner cubes of the larger wooden cube will have 3 painted faces, because:
- Each face of the larger wooden cube touches only one face of a corner cube.
- As the larger wooden cube has 3 dimensions, each corner will touch 3 of its faces.

Therefore, we can first count the number of corner cubes, which is:
- A cube has 8 corners.
  - There are 8 smaller cubes with 3 painted faces.

Moving on, we can compute the probability as:

$$P(3\text{ painted faces}) = \frac{\text{Number of corner cubes}}{\text{Total number of smaller cubes}}$$

Given:
- Number of corner cubes = 8
- Total number of smaller cubes = 27

Which is:

$$P(3\text{ painted faces}) = \frac{8}{27} \approx 0.2963$$

The probability that a randomly chosen smaller cube has 3 painted faces is approximately 29.63%.

# 2 Problem B

## 2.1 Question

In a penalty shootout, two footballers, Player X and Player Y, are known for their precision. Player X has an 85% probability of scoring a goal, while Player Y has a 80% probability. If both players take a shot one after the other, what is the probability that at least one of them scores?

### 2.2 Answer

I believe that we can use complementary probability to solve this, which is:

$$P(\text{at least one scores}) = 1 - P(\text{neither scores})$$

where $P(\text{at least one scores})$ is the probability that Player X misses and Player Y misses.

- The probability that Player X misses is $1 - 0.85 = 0.15$.
- The probability that Player Y misses is $1 - 0.80 = 0.20$.

We can then compute the probability that neither Player X nor Player Y scores as:

$$P(\text{neither scores}) = P(\text{X misses}) * P(\text{X misses}) = 0.15 * 0.20$$

Then we can have:

$$P(\text{at least one scores}) = 1 - (0.15 * 0.20) = 0.97$$

The probability that at least one of the two players scores is 97%.

# 3 Problem C

## 3.1 Question

In a carnival game, players throw balls at a wall with five differently shaped targets (Please note that a throw will either hit one of the shapes or miss completely, there is no other possiblity). The probabilities of not hitting each of these shapes are:

- Star: 0.68
- Triangle: 0.95
- Square: 0.90
- Circle: 0.80
- Pentagon: 0.81

Given that the outcomes for each shape are independent, what is the probability that a player's shot does not hit any of the shapes?

### 3.2 Answer

Essentially, if the outcomes for each shape are independent, then the probability that a shot misses all the shapes is the product of the probabilities of not hitting each shape individually.

Given:

$$P(\text{Star}) = 0.68$$

$$P(\text{Triangle}) = 0.95$$

$$P(\text{Square}) = 0.90$$

$$P(\text{Circle}) = 0.80$$

$$P(\text{Pentagon}) = 0.81$$

The probability of not hitting any shape is given by:

$$P(\text{miss all}) = P(\text{Star}) * P(\text{Triangle}) * P(\text{Square}) * P(\text{Circle}) * P(\text{Pentagon}) \approx 0.3761$$

The probability that a player's shot does not hit any of the shapes is approximately 37.67%.

## 4 Problem D

### 4.1 Question

Premise:

You and your friend are sitting in the CDS building waiting for your next class. Your friend is bored and suggests playing a game where you roll two six-sided dice, You win if the sum of the two dice equals 7.

### 4.2 Part a

If you decide to play the game just once, what is the probability that you win?

### 4.3 Answer

Essentially, we need to first determine all the ways we can get a sum of 7.

So, we can have the following list to see all that:

1. $\text{Die}_1$ = 1, $\text{Die}_2$ = 6
2. $\text{Die}_1$ = 2, $\text{Die}_2$ = 5
3. $\text{Die}_1$ = 3, $\text{Die}_2$ = 4
4. $\text{Die}_1$ = 4, $\text{Die}_2$ = 3
5. $\text{Die}_1$ = 5, $\text{Die}_2$ = 2
6. $\text{Die}_1$ = 6, $\text{Die}_2$ = 1

As we can tell from the list, there are 6 possible combinations that will give us a sum of 7.

The total number of outcomes when rolling two dice is

$$6 * 6 = 36$$

since each die has 6 sides and they are independent of each other.

Therefore, the probability P of the sum being 7 can be given by:

$$P(\text{sum} = 7) = \frac{\text{number of favorable outcome}}{\text{total number of outcomes}}$$

which we plug in the numbers:

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$$

So, if we can play the game only once, the probability that we win is $\frac{1}{6}$.

### 4.4 Part b

If you play the game three times in a row and don't remember your previous rolls, what is the probability you win at least once?

## 4.5 Answer

This could be a tough one. So, I would imagine it might be easier to first calculate the probability that we lose all three times and then subtract that from 1.

Therefore, this would give us to start with calculate the probability that we lose in one try:

$$P(\text{not sum} = 7) = 1 - P(\text{sum} = 7) = 1 - \frac{1}{6} = \frac{5}{6}$$

Next would be to calculate the probability that we lose all three times:

Given that each roll is independent, we have:

$$P(\text{lose 3 times}) = \left(\frac{5}{6}\right)^3$$

Now, we can calculate the probability that we win at least once. This is the complement of losing all three times, which is:

$$P(\text{win at least once}) = 1 - P(\text{lose 3 times}) = 1 - \left(\frac{5}{6}\right)^3 \approx 0.4213$$

Therefore, the probability that we can win at least once is approximately 42.13% if we play the game three times in a row.

## 4.6 Part c

If you play 4 games, what is the probability you win exactly twice, given the probability of winning a single game as found in Part a?

### 4.7 Answer

To solve this, I believe the binomial probability formula could be pretty helpful. So essentially, the probability of a certain number of successes in a fixed number of Bernoulli trials is given by:

$$P(X = k) = \binom{n}{x} p^{k(1-p)^{n-k}}$$

Given:

$$n = 4, k = 2, p = \frac{1}{6}$$

We can have:

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(1 - \frac{1}{6}\right)^{4-2}$$

$$P(X = 2) = 6 * \frac{1}{36} * \frac{25}{36}$$

$$P(X = 2) = \frac{75}{648}$$

$$P(X = 2) \approx 0.1157$$

So, if we play the game four times, the probability that you win exactly twice is approximately 11.57%.

## 4.8 Part d

What is the expected number of games you will play before you win?

### 4.9 Answer

So for this, we can apply the concept of geometric distribution, which gives us the probability of the first sucess on the $k^{\text{th}}$ trial.

If the $p$ is the probability of sucess on a single trial, the expected number of get the first success of a geometric distribution is given by:

$$E(X) = \frac{1}{p}$$

From part a, we found that the probability of winning a single game:

$$p = \frac{1}{6}$$

Plug in the value we can have:

$$E(X) = \frac{1}{\frac{1}{6}} = 6$$

Therefore, the expected number of games we will play before we win is 6 games.

### 4.10 Part e

Your friend now proposes a twist. If either dice shows a 1, you automatically lose, regardless of the sum. Calculate the probability of winning in this new scenario for a single game.

### 4.11 Answer

Essentially, we can first calculate the probability of rolling at least one 1:

- The probability of rolling a 1 on the first die and not on the second:

$$\frac{1}{6} * \frac{5}{6} = \frac{5}{36}$$

- The probability of rolling a 1 on the second die and not on the first:

$$\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$$

- The probability of rolling a 1 on both dice:

$$\frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

We can combine these probabilities:

$$P(\text{at least one 1}) = \frac{5}{36} + \frac{5}{36} + \frac{1}{36} = \frac{11}{36}$$

Given that:

- From Part a, there are 6 favorable outcomes out of 36 possible outcomes for the dice to sum to 7 without any constraints.

However, given the new rule, we need to subtract the outcomes where a 1 appears in the combinations that sum to 7, in which:

- The combinations that sum to 7 and include a 1 are: (1, 6) and (6, 1). There are 2 such combinations.

$$P(\text{sum} = 7 \text{ and no 1's}) = \frac{6-2}{36} = \frac{1}{9}$$

In a sense, we need to consider the two probabilities together, and to win under this new scenario, the dice have to sum to 7 and neither die can show a 1:

$$P = P(\text{sum} = 7 \text{ and no 1's}) * (1 - P(\text{at least one 1}))$$

$$P = \frac{1}{9} * \left(1 - \frac{11}{36}\right)$$

$$P = \frac{1}{9} * \frac{25}{36}$$

$$P = \frac{25}{324}$$

$$P \approx 0.0772$$

Therefore, with the new rule in place, the probability of winning in a single game is approximately 7.72%.

## 4.12 Part f

You and your friend decide to further tweak the rules. Now, if the first dice shows a 1, you lose, but if the second dice shows a 6, you automatically win, regardless of the sum. Find the probability that you win the game in this new scenario.

### 4.13 Answer

This can also be a tough one to think. So I believe we can have:

- The second die shows a 6 (an automatic win for us)
  - Regardless of what the first die shows:
    $$P(\text{Die1 is anything}) = 1$$
  - $$P(\text{Die2} = 6) = \frac{1}{6}$$
  - Combine them together, we have:
    $$1 * \frac{1}{6} = \frac{1}{6}$$
- The sum of the dice equals 7, without the first die showing a 1 (since that would be a loss)
  - For the dice to sum to 7 and not have the first die be a 1, we can have the combinations: (2, 5), (3, 4), (4, 3), and (5, 2)
  - And one of these combinations, (5, 2), would already make us win due to the second die showing a 6. Thus, we will not count it again
  - So, there are 3 favorable combinations:

$$P(\text{sum} = 7 \text{ with new rule}) = \frac{3}{36} = \frac{1}{12}$$

We can now combine these probabilities:

$$P(\text{win with new rules}) = P(\text{Die2} = 6) + P(\text{sum} = 7 \text{ with new rule})$$

$$P(\text{win with new rules}) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

Therefore, with the further tweaked rules, the probability of winning in a single game is 25%.

# 5 Problem E

## 5.1 Question

Imagine you're Batman and the overall rate of individuals having committed a crime (let's denote this as event A) in Gotham is P(A)

There's a certain marker, a unique tattoo (let's denote this as event B), which has been found common among many criminals. You observed that among the individuals with this tattoo, the probability of them having committed a crime is P(A | B) and this is higher than P(A)

$$P(A) = 0.4$$

$$P(B) = 0.3$$

$$P(A \mid B) = 0.5$$

Using the given probabilities and the definition of conditional probability, determine the exact value of $P(B|A)$. Compare it to the value of $P(B)$, is there any relation you find between $P(B|A)$ and $P(B)$ (Greater than, Lesser than, Equal to)

### 5.2 Answer

We know that:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Given:

$$P(A) = 0.4$$

$$P(B) = 0.3$$

$$P(A \mid B) = 0.5$$

We want to find:

$$P(B \mid A)$$

in which:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

We can get the value of $P(A \cap B)$ from our given data.

From:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A \mid B) * P(B) = 0.5 * 0.3 = 0.15$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.15}{0.4} = 0.375$$

Compare

$$P(B|A)$$

with

$$P(B)$$

$$P(B \mid A) = 0.375$$

$$P(B) = 0.3$$

$$\therefore P(B \mid A) > P(B)$$

Therefore, given that an individual has committed a crime (event A), the probability that they have the unique tattoo (event B) is greater than the general probability of having the unique tattoo in the population.

# 6 Problem G

## 6.1 Question

Consider a radioactive source emitting alpha particles at an average rate of 4 particles per second. What is the probability that in a particular one-second interval, less than two particles are emitted?

## 6.2 Answer

The situation described can be modeled by a Poisson distribution. The Poisson distribution gives the probability of a given number of events happening in a fixed interval of time or space, given a fixed average rate of occurrence.

We want to find the probability that in a particular one-second interval, less than two particles are emitted. This means either 0 or 1 particle is emitted:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

So now, we can substitute into the Poisson formula for k = 0 and k = 1, and given λ = 4:

$$P(X = 0) = \frac{4^0 e^{-4}}{0!}$$

$$P(X = 1) = \frac{4^1 e^{-4}}{1!}$$

Then we can have:

$$P(X = 2) = P(X = 0) + P(X = 1) \approx 0.0916$$

The chance that less than two particles are emitted in a one-second interval is about 9.16%.

# 7 Problem H

## 7.1 Question

Consider the following joint probability table for two random variables, $(X)$ and $(Y)$:

| | **Y = 1** | **Y = 2** |
|---|---|---|
| **X = 1** | 0.2 | 0.3 |
| **X = 2** | 0.1 | 0.4 |

What is the marginal probability $(P(Y = 1)$?

## 7.2 Answer

To find the marginal probability $P(Y = 1)$, we can sum over all possible values of $X$ for $Y = 1$.

Using the given table, the values are:

For $X = 1$ and $Y = 1$: $P(X = 1, Y = 1) = 0.2$

For $X = 2$ and $Y = 1$: $P(X = 2, Y = 1) = 0.1$

Now, we can determine $P(Y = 1)$:

$$P(Y = 1) = \sum_x P(X = x, Y = 1) = 0.2 + 0.1 = 0.3$$

Hence, $P(Y = 1)$ = 0.3.