DS 122 Homework 6

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1 Q1 Bayesian Update with Coin Flips

In the lecture, you learned about the Bayesian update using bowls and dice as examples. Now, let's apply this knowledge to a practical scenario involving a coin.

You have a coin whose fairness you're uncertain about. You believe it could be biased towards heads with probabilities p of 0.3, 0.5, or 0.7. You decide to use Bayes' theorem to update your beliefs about this coin.

You flip the coin 10 times and observe 7 heads.

1.1 Q1.1

Assuming a uniform prior (i.e., each value of p is equally likely), perform a Bayesian update to compute the posterior probabilities for p=0.3, p=0.5, and p=0.7 given the observed data. Write out the full Bayes table you used to calcute the posterior probabilities.

1.2 Solution

Calculate the likelihood of observing 7 heads in 10 flips given a particular value of p:

$$\text{Likelihood} = \binom{10}{7} \cdot p^7 \cdot (1-p)^3$$

1. Prior: Since it's a uniform prior, the prior probability for each value of p is $\frac{1}{3}$
2. Likelihood:
   - For p = 0.3: $\binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3$
   - For p = 0.5: $\binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3$
   - For p = 0.7: $\binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3$
3. Prior × Likelihood: Multiply the prior by the likelihood for each value of p
4. Normalization constant: Sum the values in the Prior × Likelihood column.
5. Posterior: Divide each value in the Prior × Likelihood column by the normalization constant to obtain the posterior probabilities for each value of p

Bayes table to compute the posterior probabilities for $p = 0.3, 0.5, \text{and}, 0.7$.

p	**Prior**	**Likelihood**	**Prior · Likelihood**	**Posterior**
0.3	$\frac{1}{3}$	$\binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3$	$\frac{1}{3} \cdot \binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3$	$\frac{\frac{1}{3} \cdot \binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3}{\text{Norm}}$
0.5	$\frac{1}{3}$	$\binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3$	$\frac{1}{3} \cdot \binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3$	$\frac{\frac{1}{3} \cdot \binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3}{\text{Norm}}$
0.7	$\frac{1}{3}$	$\binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3$	$\frac{1}{3} \cdot \binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3$	$\frac{\frac{1}{3} \cdot \binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3}{\text{Norm}}$

p	Prior	Likelihood	Prior · Likelihood	Posterior
0.3	$\frac{1}{3}$	0.00900169	0.00300056	0.0229041
0.5	$\frac{1}{3}$	0.1171875	0.0390625	0.298174
0.7	$\frac{1}{3}$	0.26682793	0.08894264	0.678922

1.3 Q1.2

Let's consider a non-uniform prior. You have reasons to believe that the coin being fair (i.e., p=0.5) is more probable than the other two possibilities. Specifically, let the prior probabilities be:

- $P(p = 0.3) = 0.2$
- $P(p = 0.5) = 0.6$
- $P(p = 0.7) = 0.2$

Using these priors and the observed data, compute the posterior probabilities for p=0.3, p=0.5, and p=0.7. Again, write out the full Bayes table you used to calcute the posterior probabilities.

1.4 Solution

p	Prior	Likelihood	Prior · Likelihood	Posterior
0.3	0.2	$\binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3$	$0.2 \cdot \binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3$	$\frac{0.2 \cdot \binom{10}{7} \cdot (0.3)^7 \cdot (1-0.3)^3}{\text{Norm}}$
0.5	0.6	$\binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3$	$0.6 \cdot \binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3$	$\frac{0.6 \cdot \binom{10}{7} \cdot (0.5)^7 \cdot (1-0.5)^3}{\text{Norm}}$
0.7	0.2	$\binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3$	$0.2 \cdot \binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3$	$\frac{0.2 \cdot \binom{10}{7} \cdot (0.7)^7 \cdot (1-0.7)^3}{\text{Norm}}$

p	Prior	Likelihood	Prior · Likelihood	Posterior
0.3	0.2	0.00900169	0.00180034	0.0143478
0.5	0.6	0.1171875	0.0703125	0.560355

0.7	0.2	0.26682793	0.05336559	0.425297

1.5 Q1.3

Reflect on the results. How did the non-uniform prior influence the posterior beliefs about the coin's fairness? Under what circumstances might it be beneficial to use such a prior?

1.6 Solution

The non-uniform prior influenced the posterior beliefs about the coin's fairness by weighting the likelihood of the coin being fair (i.e., $p = 0.5$) more heavily compared to the uniform prior scenario. In the non-uniform prior case, a higher prior probability was assigned to $p = 0.5$ which, in turn, influenced the posterior probability to be higher for $p = 0.5$ as well, assuming other factors remain constant.

Might be beneficial in scenarios where there is some prior knowledge or strong beliefs about the parameters being estimated.

- Historical data: When historical data suggests a particular distribution of the parameter, a non-uniform prior reflective of this distribution can be a sensible choice
- Regulatory guidelines: In some fields, regulatory or safety guidelines might necessitate conservative estimates, which can be facilitated by choosing appropriate non-uniform priors

2 Q2 Imagine you have two dice

You are given two dice:

- A 6-sided die with each face showing numbers from 1 through 6 (i.e. all numbers 1-6 are equally likely).
- 4-sided tetrahedron die with faces showing numbers from 1 through 4 (i.e. all numbers 1-4 are equally likely).

2.1 Q2.1

Calculate the distribution of outcomes if there is a 75% chance of picking the 6-sided die and a 25% chance of picking the 4-sided die. In other words, compute the "mixed" distribution of outcomes (numbers you might roll) by combining the distributions of the two dice.

2.2 Solution

The probabilities of outcomes for each die:

For the 6-sided die:

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

For the 4-sided die:

$$P(1) = P(2) = P(3) = P(4) = \frac{1}{4}$$

The "mixed" probabilities for each outcome:

With the law of total probability, the mixed probability for an outcome x is:

$$P(x) = P(x|\,\text{6-sided die}) \times P(\text{6-sided die}) + P(x|\,\text{4-sided die}) \times P(\text{4-sided die})$$

Given:

- $P(\text{6-sided die}) = 0.75$
- $P(\text{4-sided die}) = 0.25$

We can have:

For × = 1:

$$P(1) = \frac{1}{6} \cdot 0.75 + \frac{1}{4} \cdot 0.25$$

$$P(1) = 0.125 + 0.0625 = 0.1875$$

For × = 2:

$$P(2) = \frac{1}{6} \cdot 0.75 + \frac{1}{4} \cdot 0.25$$

$$P(2) = 0.125 + 0.0625 = 0.1875$$

For × = 3:

$$P(3) = \frac{1}{6} \cdot 0.75 + \frac{1}{4} \cdot 0.25$$

$$P(3) = 0.125 + 0.0625 = 0.1875$$

For × = 4:

$$P(4) = \frac{1}{6} \cdot 0.75 + \frac{1}{4} \cdot 0.25$$

$$P(4) = 0.125 + 0.0625 = 0.1875$$

For x = 5:

$$P(5) = \frac{1}{6} \cdot 0.75 + 0 \cdot 0.25$$

$$P(5) = 0.125$$

For x = 6:

$$P(5) = \frac{1}{6} \cdot 0.75 + 0 \cdot 0.25$$

$$P(5) = 0.125$$

Final Probabilities:

1. P(1) = 0.1875
2. P(2) = 0.1875
3. P(3) = 0.1875
4. P(4) = 0.1875
5. P(5) = 0.125
6. P(6) = 0.125

2.3 Q2.2

Calculate the probability of superiority of this mixed distribution over a single roll of a four-sided die. In other words, what is the probability that the outcome of the mixture of distributions calculated in part A exceeds the outcome of a single 4-sided die roll.

2.4 Solution

For a 4-sided die roll of 1:

We want to calculate $P(\text{Mixed Distribution} > 1)$.

This is equivalent to the probability of rolling a 2, 3, 4, 5, or 6 from the mixed distribution.

$$P(\text{Mixed} > 1) = P(2) + P(3) + P(4) + P(5) + P(6)$$

$$P(\text{Mixed} > 1) = 0.1875 + 0.1875 + 0.1875 + 0.125 + 0.125 = 0.8125$$

For a 4-sided die roll of 2:

We want to calculate $P(\text{Mixed Distribution} > 2)$.

This is the probability of rolling a 3, 4, 5, or 6 from the mixed distribution.

$$P(\text{Mixed} > 2) = P(3) + P(4) + P(5) + P(6)$$

$$P(\text{Mixed} > 2) = 0.1875 + 0.1875 + 0.125 + 0.125 = 0.625$$

For a 4-sided die roll of 3:

We want to calculate $P(\text{Mixed Distribution} > 3)$.

This is the probability of rolling a 3, 4, 5, or 6 from the mixed distribution.

$$P(\text{Mixed} > 2) = P(4) + P(5) + P(6)$$

$$P(\text{Mixed} > 2) = 0.1875 + 0.125 + 0.125 = 0.4375$$

For a 4-sided die roll of 4:

We want to calculate $P(\text{Mixed Distribution} > 3)$.

This is the probability of rolling a 3, 4, 5, or 6 from the mixed distribution.

$$P(\text{Mixed} > 2) = P(5) + P(6)$$

$$P(\text{Mixed} > 2) = 0.125 + 0.125 = 0.25$$

We now average the probabilities calculated above, weighted by the probability of each outcome on the 4-sided die (which is $\frac{1}{4}$ or each outcome).

$$P(\text{Superiority}) = \times 0.8125 + 4 \times 0.625 + \times 0.4375 + \times 0.25$$

$$P(\text{Superiority}) = 0.203125 + 0.15625 + 0.109375 + 0.0625 = 0.53125$$

Therefore, the probability that the outcome of the mixed distribution exceeds the outcome of a single 4-sided die roll is 53.125%.

2.5 Q2.3

Reflect on the results. How does the mixed distribution of the two dice influence the likelihood of rolling specific numbers compared to the individual distributions of each die?

2.6 Solution

The weighting of 75% for the 6-sided die and 25% for the 4-sided die greatly impacts the mixed distribution. If the dice were chosen with equal probability, the distribution would look different, with a heavier influence from the 4-sided die. And also, the mixed distribution has an overall 53.125% chance of producing a number greater than a roll from a 4-sided die. This superiority is influenced by both the extended range of the mixed distribution (possible rolls of 5 and 6) and the enriched middle range.

3 Q3 Computational

Download the notebook from this link https://piazza.com/class_profile/get_resource/llqwp5rdfue104/lo64p9juazs765 and upload the same post completion. CSV File: https://piazza.com/class_profile/get_resource/llqwp5rdfue104/lo6geuamxwn318