DS 122 Homework 7

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Contents

1 Q1 Bayes Factor Analysis of a Coin Toss

You have come across a mysterious coin and are attempting to determine whether it is fair or biased. You decide to use Bayes factors to assist in your analysis.

Background

A Bayes factor (BF) measures the strength of evidence for one hypothesis over another. Specifically, it's the ratio of the likelihoods for two hypotheses.

A BF > 1 indicates evidence in favor of the first hypothesis, a BF < 1 indicates evidence in favor of the second hypothesis, and a BF of 1 indicates the data does not favor one hypothesis over the other.

Data

You decide to flip the coin 100 times and observe the following:

Heads: 60 Tails: 40

1.1 Q1.1 Part A

Calculate the likelihood of the data under two hypotheses:

1. The coin is fair: Heads and Tails both have a probability of 0.5.
2. The coin is biased towards Heads with a probability of 0.6.

Compute the Bayes factor for the likelihood of a fair coin vs. the likelihood of a biased coin.

1.2 Solution

Use the binomial distribution, which gives the probability of observing a given number of successes (in this case, heads) in a fixed number of trials (coin flips), with a constant probability of success on each trial.

Given:

- For hypothesis 1 (the coin is fair), $p = 0.5$.
- For hypothesis 2 (the coin is biased towards heads), $p = 0.6$.

Likelihood of observing 60 heads in 100 flips if the coin is fair ($p = 0.5$):

$$L(\text{fair}) = \binom{100}{60} \cdot (0.5)^{60} \cdot (0.5)^{40} \approx 0.0108439$$

Likelihood of observing 60 heads in 100 flips if the coin is biased towards heads ($p = 0.6$):

$$L(\text{biased}) = \binom{100}{60} \cdot (0.6)^{60} \cdot (0.4)^{40} \approx 0.0812191$$

$$\text{BF} = \frac{L(\text{fair})}{L(\text{biased})}$$

$$\text{BF} = \frac{L(\text{fair})}{L(\text{biased})} = \frac{0.0108439}{0.0812191} \approx 0.1335$$

A Bayes factor of less than 1 indicates evidence in favor of the second hypothesis. In this case, the BF of approximately 0.1335 suggests that the data provides evidence in favor of the hypothesis that the coin is biased towards heads with a probability of 0.6, rather than being fair.

1.3 Q1.2 Part B

Given a prior belief that there is a 50% chance the coin is fair and a 50% chance the coin is biased, convert the Bayes factor into a posterior probability for each hypothesis.

What can you conclude about the fairness of the coin given the data and your prior beliefs?

1.4 Solution

Let's denote:

- H_f as the hypothesis that the coin is fair,
- H_b as the hypothesis that the coin is biased,
- $P(H_f)$ as the prior probability that the coin is fair,
- $P(H_b)$ as the prior probability that the coin is biased,
- $L(H_f)$ as the likelihood of the data if the coin is fair,
- $L(H_b)$ as the likelihood of the data if the coin is biased,
- $P(H_f \mid \text{data})$ as the posterior probability that the coin is fair,
- $P(H_b \mid \text{data})$ as the posterior probability that the coin is biased.

Given:

- $P(H_f) = P(H_b) = 0.5$
- $L(H_f) = 0.0108439$
- $L(H_b) = 0.0812191$
- $\text{BF} \approx 0.1335$

The posterior odds are given by the prior odds times the Bayes factor:

$$\frac{P(H_f \mid \text{data})}{P(H_b \mid \text{data})} = \frac{P(H_f)}{P(H_b)} \cdot \text{BF}$$

Since the prior odds are 1 (because $(P(H_f) = P(H_b))$), the posterior odds are simply the Bayes factor:

$$\frac{P(H_f \mid \text{data})}{P(H_b \mid \text{data})} = \text{BF}$$

Normalize these odds to find the posterior probabilities:

$$P(H_f \mid \text{data}) = \frac{\text{BF}}{\text{BF} + 1}$$

$$P(H_b \mid \text{data}) = \frac{1}{\text{BF} + 1}$$

Calculate these posterior probabilities:

For the fair coin hypothesis H_f:

$$P(H_f \mid \text{data}) = \frac{\text{BF}}{\text{BF} + 1}$$

$$P(H_f \mid \text{data}) = \frac{0.1335}{0.1335 + 1} = \frac{0.1335}{1.1335} \approx 0.1178$$

For the biased coin hypothesis (H_b):

$$P(H_b \mid \text{data}) = \frac{1}{\text{BF} + 1}$$

$$P(H_b \mid \text{data}) = \frac{1}{0.1335 + 1} = \frac{1}{1.1335} \approx 0.8822$$

These probabilities sum to 1, as they should. Given the data and the prior beliefs, the posterior probability strongly favors the hypothesis that the coin is biased towards heads $(P(H_b \mid \text{data}) \approx 88.22\%$ over the hypothesis that the coin is fair $(P(H_f \mid \text{data}) \approx 11.78\%$.

This conclusion is based on the observed data of 60 heads in 100 flips and the prior belief that there was an equal chance of the coin being fair or biased.

2 Q2 Computational

Please complete the computational notebook and upload the same.

- Notebook: https://piazza.com/class_profile/get_resource/llqwp5rdfue104/log93npy2807mf
- Birds dataset: https://piazza.com/class_profile/get_resource/llqwp5rdfue104/log93nkqxf57m1