Parameter Estimation: Maximum Likelihood Estimation

- The maximum likelihood estimate (MLE) is that value of the parameter θ that maximizes the probability of the data as a function of that parameter (i.e., the likelihood).

- Steps to find MLE $\hat{\theta}$ for distribution parameter θ:

    1. Compute the likehood function $p\left(X;\theta\right)$.
    2. Compute the corresponding log-likelihood function $\log p\left(X;\theta\right)$.
    3. Take the derivative of the log-likelihood function w.r.t. with respect to θ, $\frac{\partial \log p(X;\theta)}{\partial \theta}$.
    4. Find the MLE by setting the derivative to zero: $\frac{\partial \log p(X;\theta)}{\partial \theta} = 0$.
    5. (Optional) Confirm that the obtained critical point is indeed a maximum by taking the second derivative or from the plot of the log-likelihood function.

Some properties of the natural logarithm:

Product rule	$\log ab = \log a + \log b$
Quotient rule	$\log \frac{a}{b} = \log a - \log b$
Power rule	$\log a^n = n \log a$
Exponential\logarithmic	$\log e^x = e^{\log x} = x$

Some properties of the derivatives:

Natural logarithm	$\frac{d}{dx} \log x = \frac{1}{x}$
Product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\ du/dx - u\ dv/dx}{v^2}$
Chain rule	$\frac{d}{dx} z(y(x)) = \frac{dz}{dy}\frac{dy}{dx}$

Problem A

Willem-Alexander spins a coin three times and observes no heads. He then gives the coin to Amalia. She spins it until the first head occurs, and ends up spinning it four times total. Let θ denote the probability the coin comes up heads.

a. What is the likelihood of θ?
b. What is the MLE of θ?